# $A_\infty$ STRUCTURES

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Abstract. We discuss $\mathcal{A}_\infty$ algebras and modules, tensor products of $\mathcal{A}_\infty$ modules, and type $D$ structures on graded $\mathbf{k}$-modules.

Contents



## 1. Introduction

These lecture notes follow sections 2.1, 2.2, and 2.3 of *Bordered Heegaard Floer homology* by Lipshitz, Ozsváth, and Thurston [1]. The first two sections of these lecture notes focus on the definitions of $\mathcal{A}_\infty$ algebras, $\mathcal{A}_\infty$ modules, and the $\mathcal{A}_\infty$ tensor product. The third section of these notes defines type $D$ structures and proves a correspondence lemma that associates $D$ homomorphisms with differential graded (dg) homomorphisms.

## 2. $\mathcal{A}_\infty$ algebras and modules

We work in the category of $\mathbb{Z}$-graded complexes over a fixed commutative ground ring $k$, where $k$ has has characteristic 2. The symbol $\otimes$ with no subscript will be implicitly understood to be the tensor product over the ring $k$. We consider modules $M$ over $k$, where

$$M = \bigoplus_{d\in\mathbb{Z}} M_d.$$

Also for $n \in \mathbb{Z}$ and $M$ a graded $k$-module, we can shift the grading of $M$ by $n$ to define $M[n]$, where $M[n]_d = M_{d-n}$.

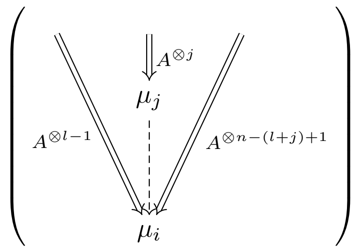
**Definition 2.1.** An $\mathcal{A}_\infty$ algebra $A$ over $k$ is a graded $k$-module equipped with $k$-linear multiplication maps $\mu_i : A^{\otimes i} \to A[2-i]$ satisfying the compatibility conditions

$$\sum_{i+j=n+1} \sum_{l=1}^{n-j+1} \mu_i(a_1 \otimes \ldots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \ldots \otimes a_{l+j-1}) \otimes a_{l+j} \otimes \ldots \otimes a_n)$$

for each $n \geq 1$.

We can visualize the compatibility conditions of $\mathcal{A}_\infty$ modules in the following diagram:

$$0 = \sum_{i+j=n+1} \sum_{l=1}^{n-j+1}$$




The double arrows represent elements of $A^{\otimes i}$ coming in, where $i \geq 1$. The single dotted arrow represents an element of $A$ coming in. We will use this diagram style throughout these lecture notes.

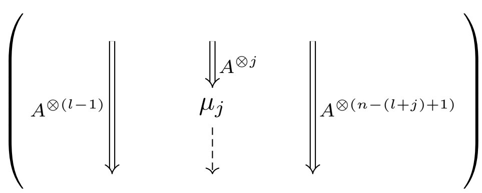
If we forget about $\mu_i$ for $i > 1$, we can view an $\mathcal{A}_\infty$ algebra as a chain complex in the usual sense. If for $(A, \{\mu_i\}_{i=1}^\infty)$, we have $\mu_i = 0$ for $i > 2$, then we call $(A, \{\mu_i\}_{i=1}^\infty)$ a *differential graded* (dg) algebra over $k$. $\mu_1 : A \to A[1]$ would be the differential and $\mu_2 : A \otimes A \to A$ would be the multiplication map.

We can reinterpret the compatibility conditions of an $\mathcal{A}_\infty$ algebra if we define

$$\mathcal{T}^*(A[1]) := \bigoplus_{n=0}^{\infty} A^{\otimes n}[n].$$

This way, we can combine the $\mu_i$'s into a single map $\mu : \mathcal{T}^*(A[1]) \to A[2]$ and define $\overline{D} : \mathcal{T}^*(A[1]) \to \mathcal{T}^*(A[1])$ by

$$\overline{D} := \sum_{j=1}^{n} \sum_{l=1}^{n-j+1}$$




The compatibility condition is equivalent to $\mu \circ \overline{D} = 0$, which is also equivalent to $\overline{D} \circ \overline{D} = 0$.

**Definition 2.2.** An $\mathcal{A}_\infty$ algebra $(A, \{\mu_i\}_{i=1}^\infty)$ is *operationally bounded* if $\mu_i = 0$ for $i$ sufficiently large.

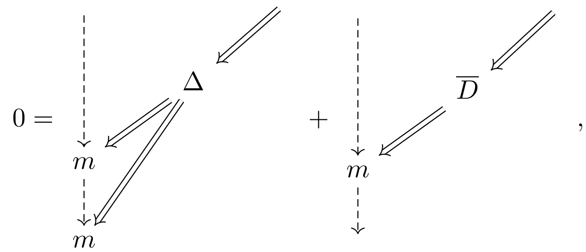
**Definition 2.3.** A (right) $\mathcal{A}_\infty$ module $\mathcal{M}$ over $\mathcal{A}$ is a graded $k$-module $M$ equipped with operations

$$m_i : M \otimes A^{\otimes(i-1)} \to M[2-i]$$

satisfying the compatibility conditions

$$\begin{aligned} 0 = &\sum_{i+j=n+1} m_i(m_j(x \otimes a_1 \otimes \ldots \otimes a_{j-1}) \otimes \ldots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} \sum_{l=1}^{n-j} m_i(x \otimes a_1 \otimes \ldots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \ldots \otimes a_{l+j-1}) \otimes \ldots a_{n-1}), \end{aligned}$$

which can be visualized as




where $\Delta : \mathcal{T}^*(A[1]) \to \mathcal{T}^*(A[1]) \otimes \mathcal{T}^*(A[1])$ is the map

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{m=0}^{n} (a_1 \otimes \ldots \otimes a_m) \otimes (a_{m+1} \otimes \ldots \otimes a_n).$$

**Definition 2.4.** An $\mathcal{A}_\infty$ module $\mathcal{M}$ over a strictly unital $\mathcal{A}_\infty$ algebra is *strictly unital* if $m_2(x \otimes 1) = x$, $m_i(x \otimes a_1 \otimes \ldots \otimes a_{i-1}) = 0$ if $i > 2$ and some $a_j = 1$. $\mathcal{M}$ is *bounded* if $m_i = 0$ for sufficiently large $i$.

We can combine the $\mu_i$ and $m_i$ into a degree 1 map

$$\overline{m} : M \otimes \mathcal{T}^*(A[1]) \to M \otimes \mathcal{T}^*(A[1])$$

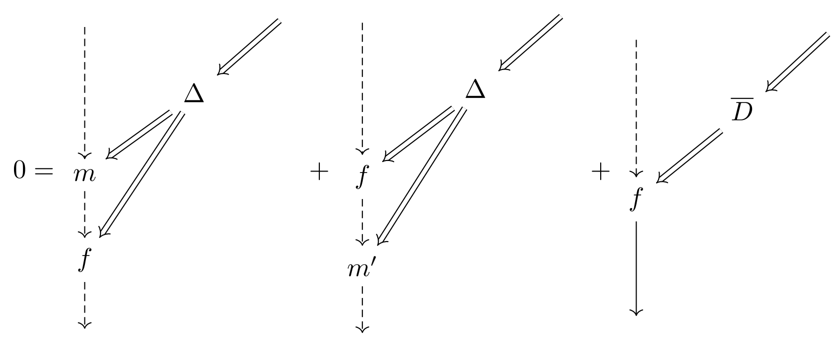
defined by

$$\begin{aligned} &\overline{m}(x_1 \otimes a_2 \otimes \ldots \otimes a_n) \\ &= \sum_{l=1}^{n} m_l(x \otimes a_2 \otimes \ldots \otimes a_l) \otimes a_{l+1} \otimes \ldots \otimes a_n \\ &+ \sum_{j=1}^{n} \sum_{l=1}^{n-j+1} x \otimes a_1 \otimes \ldots \otimes a_{l-1} \otimes \mu_j(a_l \otimes \otimes a_{l+j-1}) \otimes a_{l+j} \otimes \ldots \otimes a_n. \end{aligned}$$

**Definition 2.5.** A *strictly unital homomorphism* $f : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{A}_\infty$ modules is a collection of maps

$$f_i : M \otimes A^{\otimes(i-1)} \to M'[1-i]$$

satisfying



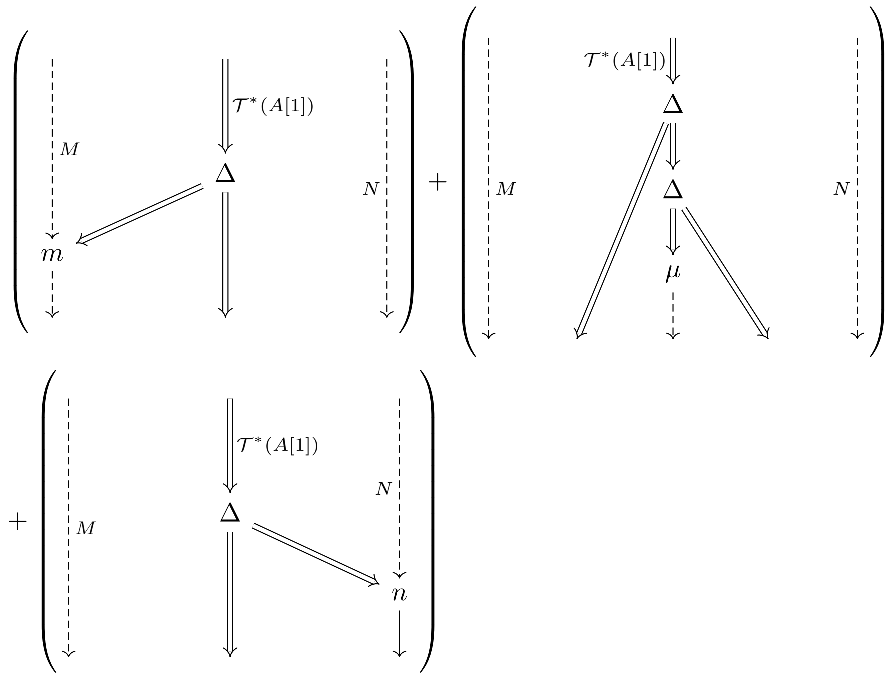

One example of a strictly unital homomorphism is the *identity homomorphism*, defined by $\mathbb{1} : \mathcal{M} \to M$, $\mathbb{1}_1(x) = x$, $\mathbb{1}_i(x \otimes A^{\otimes(i-1)}) = 0$ for $i > 1$. In the Bordered Heegaard Floer homology paper, compsition of homomorphisms of $\mathcal{A}_\infty$ modules is defined, and in this sense, $\mathbb{1}$ acts as the identity homomorphism.

## 3. $\mathcal{A}_\infty$ TENSOR PRODUCTS

**Definition 3.1.** Let $\mathcal{M}$ be a right $\mathcal{A}_\infty$ module and let $\mathcal{N}$ be a left $\mathcal{A}_\infty$ module over an $\mathcal{A}_\infty$ algebra $\mathcal{A}$. Their $\mathcal{A}_\infty$ *tensor product* is the chain complex

$$\mathcal{M}\widetilde{\otimes}_{\mathcal{A}}\mathcal{N} := M \otimes \mathcal{T}^*(A[1]) \otimes N$$

equipped with the boundary operator




An important case is when $\mathcal{N} = \mathcal{A}$, in which we have

$$\overline{\mathcal{M}} = \mathcal{M}\widetilde{\otimes}_{\mathcal{A}}\mathcal{N} := M \otimes \mathcal{T}^*(A[1]) \otimes \mathcal{A},$$

which has the structure of an $\mathcal{A}_\infty$ module over $\mathcal{A}$.

In the lecture notes on Bordered Heegaard Floer homology, homotopies between homomorphisms of $\mathcal{A}_\infty$ modules is defined. Using these definitions, the authors prove the following proposition:

**Proposition 3.2.** *If $\mathcal{A}$ and $\mathcal{M}$ are strictly unital, $\overline{\mathcal{M}}$ is homotopy equivalent to $\mathcal{M}$.*

## 4. TYPE $D$ STRUCTURES

**Definition 4.1.** Fix a dg algebra $\mathcal{A}$. Let $N$ be a graded $k$-module, equipped with a map

$$\delta^1_N : N \to (A \otimes N)[1]$$

such that

$$(\mu_1 \otimes \mathbb{1}_N) \circ (\mathbb{1}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{1}_N) \circ \delta^1 : N \to A \otimes N$$

vanishes. We call $(N, \delta^1_N)$ a *type $D$ structure over $A$ with base ring $k$.* A $k$-module map $\psi : N_1 \to A \otimes N_2$ is a $D$-structure homomorphism if

$$(\mu_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1) \circ \delta^1_{N_1} + (\mu_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \delta^1_{N_2}) \circ \psi^1 + (\mu_1 \otimes \mathbb{1}_{N_2}) \circ \psi^1 = 0.$$

**Lemma 4.2.** *If $(N, \delta^1)$ is a type $D$ structure, then $A \otimes N$ can be given the structure of a left $A$ module, with*

$$\begin{aligned} m_1(a \otimes y) &= [(\mu_2 \otimes \mathbb{1}_N) \circ (\mathbb{1}_A \otimes \delta^1) + \mu_1 \otimes \mathbb{1}_N] \\ m_2(a_1 \otimes (a \otimes y)) &= \mu_2(a_1 \otimes a) \otimes y. \end{aligned}$$

*Moreover, if $\psi^1 : N_1 \to A \otimes N_2$ is a $D$-structure homomorphism, then there is an induced map of dg modules from $A \otimes N_1$ to $A \otimes N_2$ defined by $(m_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1)$.*

So by the lemma, we can associate a dg module $\mathcal{N}$ to a type $D$ structure $(N, \delta^1)$.

**Lemma 4.3.** *Let $N_1$, $N_2$ be two type $D$ structures over a dg algebra $A$, and $\mathcal{N}_1 := A \otimes N_1, \mathcal{N}_2 := A \otimes N_2$ be their associated dg modules. The correspondence from the previous lemma gives an isomorphism between the space of type $D$ homomorphisms $N_1 \to A \otimes N_2$ with the sapce of dg homomrphisms from $\mathcal{N}_1$ to $\mathcal{N}_2$. Two type $D$ homomprhisms are homotopic iff the corresponding dg homomrphisms are homotopic by an $A$-equivariant homotopy.*

*Proof.* We only prove the correspondence between dg homomorphisms and type $D$ homomorphisms. Given $\psi : \mathcal{N}_1 \to \mathcal{N}_2$ a dg homomorphism, we prove $\psi^1 : N_1 \to A \otimes N_2$, $\psi^1(x) = \psi(1 \otimes x)$ is a $D$ structure homomorphism. Indeed,

$$\begin{aligned} &(\mu_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1) \circ \delta^1_{N_1} + (\mu_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \delta^1_{N_2}) \circ \psi^1 + (\mu_1 \otimes \mathbb{1}_{N_2}) \circ \psi^1 \\ =&m'_2 \circ (\mathbb{1}_A \circ \psi^1) \circ \delta^1_{N_1} + m'_1 \circ \psi^1 \\ =&\psi \circ \delta^1_{N_1} + \psi \circ \delta^1_{N_1} \\ =&0 \end{aligned}$$

because we have

$$\begin{aligned} m'_2 \circ (a \otimes \psi^1(x)) =& m'_2(a \otimes \psi(1 \otimes x)) \\ =&\psi(m_2(a \otimes (1 \otimes x))) \\ =&\psi(\mu_2(a \otimes 1) \otimes x) \\ =&\psi(a \otimes x) \end{aligned}$$

and

$$\begin{aligned} m'_1 \circ \psi^1(x) =& m'_1 \circ \psi(1 \otimes x)) \\ =&\psi(m_1(1 \otimes x)) \\ =&\psi([(\mu_2 \otimes \mathbb{1}_{N_1}) \circ (\mathbb{1}_A \otimes \delta^1_{N_1}) + \mu_1 \otimes \mathbb{1}_N](1 \otimes x)) \\ =&\psi((\mu_2 \otimes \mathbb{1}_{N_1}) \circ (1 \otimes \delta^1_{N_1}(x)) + \mu_1(1) \otimes x) \\ =&\psi(\delta^1_{N_1}(x) + 0 \otimes x) \\ =&\psi(\delta^1_{N_1}(x)). \end{aligned}$$

Now given $\psi^1 : N_1 \to A \otimes N_2$ a $D$-structure homomorphism, we see that $m'_2 \circ (\mathbb{1}_A \otimes \psi^1)$ is a dg homomorphism from $\mathcal{N}_1 = A \otimes N_1$ to $\mathcal{N}_2 = A \otimes N_2$. Indeed, we see that $m_2' \circ (\mathbb{1}_A \otimes \psi^1)$ commutes with the algebra action:

$$\begin{aligned} m_2'(a_1 \otimes (m_2' \circ (\mathbb{1}_A \otimes \psi^1)(a_2 \otimes x))) &= m_2'(a_1 \otimes (m_2' \circ (a_2 \otimes \psi^1(x))) \\ &= m_2'(\mu_2(a_1 \otimes a_2) \otimes \psi^1(x)) \\ &= m_2' \circ (\mathbb{1}_A \otimes \psi^1)(\mu_2(a_1 \otimes a_2) \otimes x) \\ &= m_2' \circ (\mathbb{1}_A \otimes \psi^1)(m_2(a_1 \otimes (a_2 \otimes x))), \end{aligned}$$

where the second equality comes from the definition of $m_2'$ and the compatibility conditions for $A_\infty$ algebras. Second, we have that $m_2' \circ (\mathbb{1}_A \otimes \psi^1)$ commutes with the chain differential. Indeed,

$$\begin{aligned} & m_1'(m_2' \circ (\mathbb{1}_A \otimes \psi^1)(a \otimes x)) \\ = & \big[m_2' \circ (\mathbb{1}_A \otimes m_1') \circ (\mathbb{1}_A \otimes \psi^1)(a \otimes x)\big] \\ & + \big[m_2' \circ (\mu_1 \otimes \mathbb{1}_A \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1)(a \otimes x)\big] \\ = & \big[m_2' \circ (\mathbb{1}_A \otimes (m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ \delta^1))(a \otimes x)\big] \\ & + \big[m_2' \circ (\mu_1 \otimes \mathbb{1}_A \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1)(a \otimes x)\big] \\ = & \big[m_2' \circ (\mathbb{1}_A \otimes \mu_2 \otimes \mathbb{1}_{N_2}) \circ (\mathbb{1}_A \otimes \psi^1) \circ (\mathbb{1}_A \otimes \delta^1)(a \otimes x)\big] \\ & + \big[m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ (\mu_1 \otimes \mathbb{1}_{N_1})(a \otimes x)\big] \\ = & \big[m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ (\mu_2 \otimes \mathbb{1}_{N_1}) \circ (\mathbb{1}_A \otimes \delta^1)(a \otimes x)\big] \\ & + \big[m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ (\mu_1 \otimes \mathbb{1}_{N_1})(a \otimes x)\big] \\ = & m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ [(\mu_2 \otimes \mathbb{1}_{N_1}) \circ (\mathbb{1}_A \otimes \delta^1) + (\mu_1 \otimes \mathbb{1}_{N_1})](a \otimes x) \\ = & m_2'(\mathbb{1}_A \otimes \psi^1)(m_1(a \otimes x)). \end{aligned}$$

The first equality is by the compatibility condition for $\mathcal{N}_2$ applied to

$$(\mathbb{1}_A \otimes \psi^1)(a \otimes x).$$

The second equality is the observation that because $\psi^1$ is a $D$-structure homomorphism, we have

$$m_2' \circ (\mathbb{1}_A \otimes \psi^1) \circ \delta^1_{N_1} + m_1' \circ \psi^1 = 0.$$

It is straightforward to see that the association of homomorphisms

$$\psi^1 \mapsto m_2' \circ (\mathbb{1}_A \otimes \psi^1)$$

is inverse to the association

$$\psi \mapsto \psi(1 \otimes \cdot).$$

□

## References

[1] Robert Lipshitz, Peter S. Ozsvath, and Dylan P. Thurston. Bordered Heegaard Floer homology. *Mem. Amer. Math. Soc.*, 254(1216):viii+279, 2018.